Pergamon

Int. J. Solids Structures Vol. 31, No. 3, pp. 317–327, 1994

0020–7683/94 $6.00 + .00

THERMOELASTIC ANALYSIS OF A MOONEY-RIVLIN SLAB UNDER INHOMOGENEOUS SHEARING

C. E. Maneschy
Department of Mechanical Engineering, University of Pittsburgh, Pittsburgh, PA 15261, U.S.A.

(*Received* 23 *April* 1993; *in revised form* 30 *August* 1993)

Abstract—In this paper the problem of shearing an infinite slab of Mooney-Rivlin material subjected to temperature differentials across its thickness is investigated employing a generalization of the classical Mooney-Rivlin model. Here, the properties that go into characterizing this nonlinear solid are assumed to depend on the temperature. Thus, in addition to satisfying the equilibrium conditions, the energy equation is also solved assuming Fourier's law of heat conduction. Results are displayed for isothermal and temperature dependent conditions in order to discuss the effect of the temperature on the appearance of "boundary layer" type of solution.

1. INTRODUCTION

Many investigations have been performed aimed at describing the mechanical behavior of elastic solid materials under the influence of temperature variations. However, most of the work accounting for this interaction between thermal and mechanical effects, has had its foundation on the assumption of small deformations (linear theory). This is in part due to the great complexity involved in dealing with the thermo-mechanical behavior of nonlinear elastic solids. Of the few studies that have been carried out in finite thermoelasticity, one that is quite exhaustive is the investigation by Chadwick (1974). More recently, Ogden (1992) discussed the thermoelastic modeling of nonlinear solids and Maneschy *et al.* (1993) noted the presence of "boundary layers" in the solution of neo-Hookean materials subjected to inhomogeneous expansion and temperature gradients.

The study of inhomogeneous deformations in nonlinear materials has lately caught a great deal of attention. In the analysis of such problems, solutions have been found for materials following a specific constitutive theory (neo-Hookean, Mooney-Rivlin, etc.). This approach has become very common following the work of Ericksen (1954–1955), who proved that universal solutions are possible in general compressible materials only if the deformations are homogeneous. He also exhibited specific classes of nonhomogeneous deformations that are possible for incompressible materials. Thus, if we are interested in the solution of a problem dealing with inhomogeneous deformations, we have to discuss it within the context of some subclass of isotropic elastic solids. Many interesting solutions using this approach have been documented recently, but only for isothermal conditions (Carrol, 1977; Antman and Guo, 1984; Rajagopal and Wineman, 1985; Rajagopal *et al.*, 1986; Haughton, 1992; Rajagopal and Tao, 1992). An interesting feature reported in some of these works (cf. Haughton, 1992; Rajagopal and Tao, 1992), is the presence of "boundary layers" for the deformation field, in that, adjacent to the boundary the deformation is nonhomogeneous while it is essentially homogeneous in the core. This behavior is believed to be the result of the material nonlinearities incorporated in the models.

It is known that the material properties of rubber-like solids are temperature dependent. These materials shear soften or shear harden as the temperature increases or decreases, respectively. Since this dependence is observed even at low temperatures, it would seem appropriate, on studying inhomogeneous deformations in finite elasticity, to employ a constitutive equation that would take this effect into account. Moreover, it is quite possible that temperature dependent (hence deformation dependent) models would predict, if it is the case, a more pronounced "boundary layer" in the deformation field than that found under isothermal conditions. One of the objectives of this work is to study such a possibility.

In this paper we analyse the problem wherein an infinite slab of a Mooney-Rivlin material is subjected to an inhomogeneous shearing at its lower surface as well as a temperature variation across its thickness. This is an extension of the work of Rajagopal *et al.* (1986) on the isothermal shearing of a neo-Hookean slab. The constitutive model used is a generalization of the classical Mooney-Rivlin theory, in that, the material properties that characterize this model are assumed to depend on the temperature. In addition to complying with the equilibrium conditions it is ensured that the energy equation is satisfied.

2. PROPORTIONAL SHEARING OF A SLAB OF MOONEY-RIVLIN MATERIAL

Let X, Y, Z denote the coordinates of a particle in its undeformed state and x, y, z the coordinates of the same particle in its deformed position. If the slab is sheared along the plane $Z = 0$ in such a manner that there is displacement only in X-coordinate direction, the deformation can be represented by:

$$x = Xf(Z), \tag{1a}$$

$$y = Y, \tag{1b}$$

$$z = Zg(Z). \tag{1c}$$

Furthermore suppose the temperature θ is of the form:

$$\theta = \theta(Z) = \bar{\theta}(z). \tag{2}$$

The deformation gradient tensor $\mathbf{F}$ is given by:

$$\mathbf{F} = \begin{bmatrix} f & 0 & Xf' \\ 0 & 1 & 0 \\ 0 & 0 & (Zg)' \end{bmatrix}, \tag{3}$$

where the prime denotes differentiation with respect to Z. If the deformation is assumed to be isochoric ($\det \mathbf{F} = 1$), we will have:

$$(Zg)' = g + Zg' = 1/f. \tag{4}$$

The left Cauchy–Green deformation tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is:

$$\mathbf{B} = \begin{bmatrix} f^2+(Xf')^2 & 0 & Xf'/f \\ 0 & 1 & 0 \\ Xf'/f & 0 & 1/f^2 \end{bmatrix}, \tag{5}$$

with inverse:

$$\mathbf{B}^{-1} = \begin{bmatrix} 1/f^2 & 0 & -Xf'/f \\ 0 & 1 & 0 \\ -Xf'/f & 0 & f^2+(Xf')^2 \end{bmatrix}. \tag{6}$$

For incompressible isotropic hyperelastic materials, i.e. materials in which the stress is derivable from a strain energy density, it follows that:

$$\mathbf{T} = -p\mathbf{I}+2\frac{\partial W}{\partial I_1}\mathbf{B}-2\frac{\partial W}{\partial I_2}\mathbf{B}^{-1}, \tag{7}$$

where I_1 and I_2 are the principal invariants, namely $I_1 = tr\mathbf{B}$ and $I_2 = tr\mathbf{B}^{-1}$, and $p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility.

We shall consider a generalization of the Mooney-Rivlin constitutive model (Truesdell and Noll, 1965), where the shear modulus, μ, is assumed to depend on the temperature. In this case, the strain energy has the form:

$$W(I_1, I_2, \theta) = \tfrac{1}{2}(\tfrac{1}{2}+\beta)\mu(\theta)[I_1-3]+\tfrac{1}{2}(\tfrac{1}{2}-\beta)\mu(\theta)[I_2-3], \tag{8}$$

with $\mu(\theta) > 0$ and $-\frac{1}{2} \leqslant \beta \leqslant \frac{1}{2}$. The corresponding constitutive equation is:

$$\mathbf{T} = -p\mathbf{I}+(\tfrac{1}{2}+\beta)\mu(\theta)\mathbf{B}-(\tfrac{1}{2}-\beta)\mu(\theta)\mathbf{B}^{-1}, \tag{9}$$

which gives the stress components:

$$T_{XX} = -p(X, Y, Z, \theta)+(\tfrac{1}{2}+\beta)\mu(\theta)(f^2+X^2f'^2)-(\tfrac{1}{2}-\beta)\mu(\theta)\frac{1}{f^2}, \tag{10}$$

$$T_{YY} = -p(X, Y, Z, \theta)+2\mu\beta, \tag{11}$$

$$T_{ZZ} = -p(X, Y, Z, \theta)+(\tfrac{1}{2}+\beta)\mu(\theta)\frac{1}{f^2}-(\tfrac{1}{2}-\beta)\mu(\theta)(f^2+X^2f'^2), \tag{12}$$

$$T_{XZ} = \mu(\theta)Xf'/f, \tag{13}$$

$$T_{XY} = T_{YZ} = 0, \tag{14}$$

where the subscripts on T denote the orientation of the stress components.

3. EQUATIONS OF EQUILIBRIUM

We shall find it convenient to write the equilibrium equations in terms of the reference coordinates X, Y, Z. In virtue of the chain rule:

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial T_{ij}}{\partial X_p}\frac{\partial X_p}{\partial x_j} = 0, \tag{15}$$

where $\partial X_p/\partial x_j = (\mathbf{F}^{-1})_{pj}$. It follows from (3) and (4) that:

$$\mathbf{F}^{-1} = \begin{bmatrix} 1/f & 0 & -Xf' \\ 0 & 1 & 0 \\ 0 & 0 & f \end{bmatrix}. \tag{16}$$

From (16) and (15)

$$\frac{1}{f}\frac{\partial T_{XX}}{\partial X}-Xf'\frac{\partial T_{XZ}}{\partial X}+f\frac{\partial T_{XZ}}{\partial Z} = 0, \tag{17}$$

$$\frac{\partial T_{YY}}{\partial Y} = 0, \tag{18}$$

$$\frac{1}{f}\frac{\partial T_{XZ}}{\partial X}-Xf'\frac{\partial T_{ZZ}}{\partial X}+f\frac{\partial T_{ZZ}}{\partial Z} = 0. \tag{19}$$

Substituting (10)–(14) into the equations above, one finds:

$$\frac{\partial p}{\partial X} = X(2\mu\beta f'^2 + \mu' f f' + \mu f f'' - \mu f'^2), \tag{20}$$

$$\frac{\partial p}{\partial Z} = X^2\left[\frac{\mu' f'^2}{2} + \beta(\mu' f'^2 + 2\mu f' f'')\right] + G(Z), \tag{21}$$

where:

$$G(Z) \equiv \frac{\mu f'}{f^3} + (\tfrac{1}{2}+\beta)\frac{(f^2\mu' + 2\mu f')}{f^4} - (\tfrac{1}{2}-\beta)(f^2\mu' + 2\mu f f'), \tag{22}$$

and $p = p(X, Z, \theta)$.

From equation

$$\frac{\partial^2 p}{\partial X\,\partial Z} = \frac{\partial^2 p}{\partial Z\,\partial X}$$

we deduce that the function f must satisfy:

$$\mu f f''' - \mu f' f'' + \mu'(2 f f'' - f'^2) + \mu'' f f' = 0. \tag{23}$$

One interesting feature of this equation is that it is independent of the parameter β. Therefore, the solution for the displacement function f found in a Mooney-Rivlin material subjected to the deformation given by equations (1) would be the same as that for a more special case such as the neo-Hookean material ($\beta = \frac{1}{2}$). It should be observed that eqn (23) reduces to that found in Rajagopal *et al.* (1986) for isothermal conditions ($\mu' = \mu'' = 0$).

Equations (20) and (21) can be integrated, with the aid of (23), to give the pressure field:

$$p = \frac{X^2}{2}(2\mu\beta f'^2 + \mu' f f' + \mu f f'' - \mu f'^2) + \int G(Z)\,\mathrm{d}Z + \bar{C}, \tag{24}$$

where $\bar{C}$ is an arbitrary constant due to the incompressibility condition.

4. ENERGY EQUATION

Let us now turn our attention to the energy equation:

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \mathbf{T}\cdot\mathbf{L} - \operatorname{div}\mathbf{q} + \rho r, \tag{25}$$

where ρ is the density, ε is the specific internal energy, $\mathbf{L}$ is the gradient of the velocity, $\mathbf{q}$ is the heat flux vector and r is the radiant heating.

At this point it would be necessary to specify a constitutive equation for the internal energy, ε, in order to completely characterize the thermoelastic response of the material. According to Eringen (1965), one can specify constitutive relations for the Helmholtz free energy, the specific entropy and the heat flux. For the problem in discussion, it can be assumed that the entropy and the free energy depend on the invariants of the deformation tensor and the temperature:

$$\eta = \eta(I_1, I_2, \bar{\theta}); \quad \psi = \psi(I_1, I_2, \bar{\theta}), \tag{26}$$

with the specific energy being related to these quantities by the expression:

$$\varepsilon = \psi + \bar{\theta}\eta. \tag{27}$$

The heat flux is given by Fourier's law of heat conduction as:

$$\mathbf{q} = -k(\bar{\theta}) \operatorname{grad} \bar{\theta}, \tag{28}$$

where k is the thermal conductivity. Since the deformation and temperature fields are independent of time, it follows from (1), (2), (26), (27) and (28) that, on neglecting the radiant heating, the energy equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{k(\bar{\theta})\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}z}\right\} = 0. \tag{29}$$

We shall assume that the thermal conductivity is a constant. This assumption is not inconsistent with the assumption that μ is temperature dependent, only that k is less sensitive to variations in temperature. Thus (29) reduces to:

$$\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}z^2} = 0, \tag{30}$$

or

$$\bar{\theta} = C_1 z + C_2, \tag{31}$$

which, from (1c) and (2), implies:

$$\theta = C_1 Z g + C_2. \tag{32}$$

If we restrict our analysis to problems for which the plane $Z = 0$ is held at constant temperature θ_0 and the plane $Z = H$ at temperature θ_H, eqn (32) becomes:

$$\theta = \frac{(\theta_H - \theta_0)}{Hg(H)} Zg + \theta_0, \tag{33}$$

where H is the initial slab thickness and $g(H)$ is the value of the displacement function at the upper surface. Equations (4), (23) and (33) constitute a system of coupled differential equations which have to be solved for the functions f and g. This requires a choice for the shear modulus $\mu(\theta)$, which is the variable connecting those three equations. Here we assume two forms of the shear modulus, one in which it increases with temperature (shear hardening), i.e.

$$\mu(\theta) = A\theta, \tag{34}$$

and the other in which the shear modulus decreases with temperature (shear softening),

$$\mu(\theta) = B - C\theta, \tag{35}$$

where A, B and C are positive constants and θ is the absolute temperature. This linear dependence of the shear modulus on temperature has been widely used in discussing the thermoelastic modeling of rubber-like materials (Treloar, 1975). The equations above can also be written, using (33), as:

$$\mu(Z) = A\left[\frac{(\theta_H-\theta_0)}{Hg(H)}Zg+\theta_0\right], \tag{36}$$

or

$$\mu(Z) = B-C\left[\frac{(\theta_H-\theta_0)}{Hg(H)}Zg+\theta_0\right]. \tag{37}$$

Equations (36) or (37) when substituted into (23) give rise to the final form for the coupled differential equations to be solved. The numerical solution of these equations is discussed in the next section.

5. NUMERICAL SOLUTION AND DISCUSSION OF RESULTS

Define the following set of nondimensional variables:

$$\bar{Z} = \frac{Z}{H}; \quad \bar{z} = \frac{z}{H}; \quad \bar{\mu} = \frac{\mu(\bar{Z})}{\mu(0)}. \tag{38}$$

For these nondimensional quantities, it can be shown that the final system of equations is:

$$f''' = \frac{f'f''}{f} + \frac{2Q}{f(1+QZg)}\left[\frac{f'^2}{f}-f''\right] \tag{39}$$

$$Zg'+g = 1/f, \tag{40}$$

with

$$Q = \frac{H(\theta_H-\theta_0)}{h\theta_0}, \tag{41}$$

if the material shear hardens with temperature (equation 34), or:

$$Q = \frac{HC(\theta_0-\theta_H)}{h(B-C\theta_0)}, \tag{42}$$

if it shear softens as the temperature decreases (equation 35). In these equations h represents the position of the upper surface of the slab on the deformed configuration. Therefore, the temperature parameter Q is, in general, unknown. Note also that if the temperature is held constant throughout the thickness ($Q = 0$), the system (39) and (40) uncouples and the function f is found independently. The bars were dropped from the equations above for simplicity.

We shall first study the problem where the plane at $Z = 0$ is sheared by a constant value such that:

$$f(0) = K_1, \tag{43}$$

which from the incompressibility condition (40) gives:

$$g(0) = 1/K_1. \tag{44}$$

Furthermore, if the layer $Z = 1$ is held fixed:

$$f(1) = 1 \tag{45}$$

$$g(1) = h/H = 1. \tag{46}$$

Boundary condition (46) simplifies the search for the solution, since it reduces the variable Q to a known value that depends both on the material model chosen and the temperature of the slab surfaces. The coupled differential equations (39) and (40), for a given value of Q, can be expressed as:

$$f''' = f_1(f'', f', f, g, Z), \tag{47}$$

$$g' = f_2(f, g, Z); \quad 0 \leqslant Z \leqslant 1, \tag{48}$$

subjected to the boundary conditions (43)–(46). This problem can be easily transformed into an initial boundary value problem which can be solved using the Runge–Kutta method. Initial values are assumed for $f'(0)$ and $f''(0)$ and the equation is numerically integrated to $Z = 1$, and checked against the boundary conditions (45) and (46) for $f(1)$ and $g(1)$. An iterative procedure is used, namely the Newton–Raphson scheme, and this procedure is terminate when the error tolerance 10^{-7} is reached.

Let us first consider the isothermal problem. In this case $Q = 0$. The displacement function $f(Z)$ vs the nondimensional coordinate along the slab thickness is displayed in Fig. 1, for different values of the lower surface shearing constant, K_1. For $K_1 > 1$ or $K_1 < 1$ the surface $Z = 0$ is sheared inwards or outwards, respectively. The value $K_1 = 1$ corresponds to the undeformed state. Material points close to the lower surface are displaced to the right ($K_1 > 1$) or to the left ($K_1 < 1$) of a vertical line $f = 1$ corresponding to the nondeformed configuration. For material points near the plane $Z = 1$ an opposite behavior is observed. This is a consequence of the incompressibility condition. In all cases we find the deformation to be inhomogeneous everywhere.

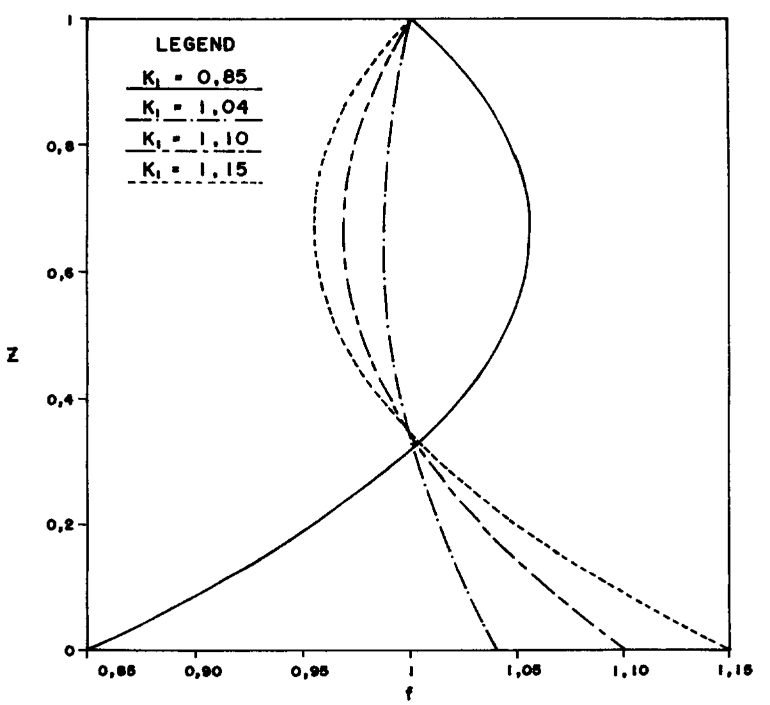

Fig. 1. Variation of f with the shearing constant K_1 for problems in which displacement is specified at the slab upper surface. ($Q = 0$.)

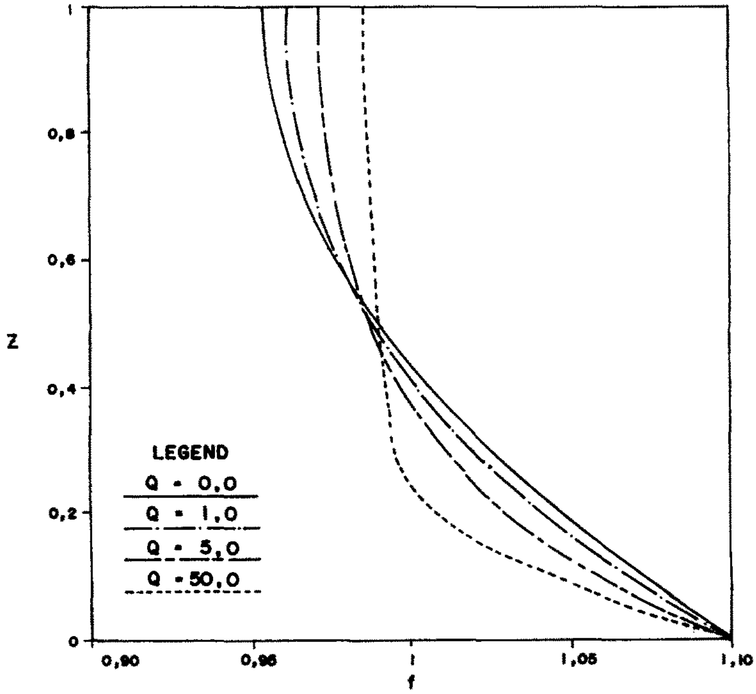

Fig. 2. Variation of f with the temperature parameter Q for problems in which displacement is specified at the slab upper surface. ($K_1 = 1.15$.)

Next we investigate the effect of the temperature on the solution while maintaining K_1 fixed. Figure 2 shows the variation of $f(Z)$ with the temperature parameter Q, for $K_1 = 1.15$. It is observed that when the value of Q increases we approach a "boundary layer" structure in which the inhomogeneity of the deformation is confined to the region close to the plane $Z = 0$. It should be noted that $Q > 0$ implies $\theta_H > \theta_0$ or $\theta_H < \theta_0$, depending on the model assumed for the shear modulus. If the material shear hardens with the temperature [model (34)], we can see that a "boundary layer" is developed adjacent to the colder boundary. However, for a shear softening material, expressed by (33), this "boundary layer" is generated adjacent to the hotter boundary. This is, of course, in keeping with our expectations.

Additionally to the dimensionless quantities given by (38), let us define the non-dimensional shear stress component as:

$$\bar{T}_{XZ} = \frac{T_{XZ}}{\mu(0)\bar{X}}; \quad \bar{X} = \frac{X}{H}. \tag{49}$$

From eqns (13) and (38), it can be shown that the stress component above reduces to:

$$T_{XZ} = \frac{(1+QZg)f'}{f}, \tag{50}$$

with the bars being again dropped for simplicity.

Figure 3 depicts the shear stress profile for different values of the temperature parameter and $K_1 = 1.15$. Negative stresses are developed at lower layers of the slab while positive stresses are observed towards the upper surface. Higher stresses are found as the temperature gradient increases.

As a second example we discuss the problem in which the shear component of the traction vector has a prescribed value at the plane $Z = 1$. For this case, the boundary condition:

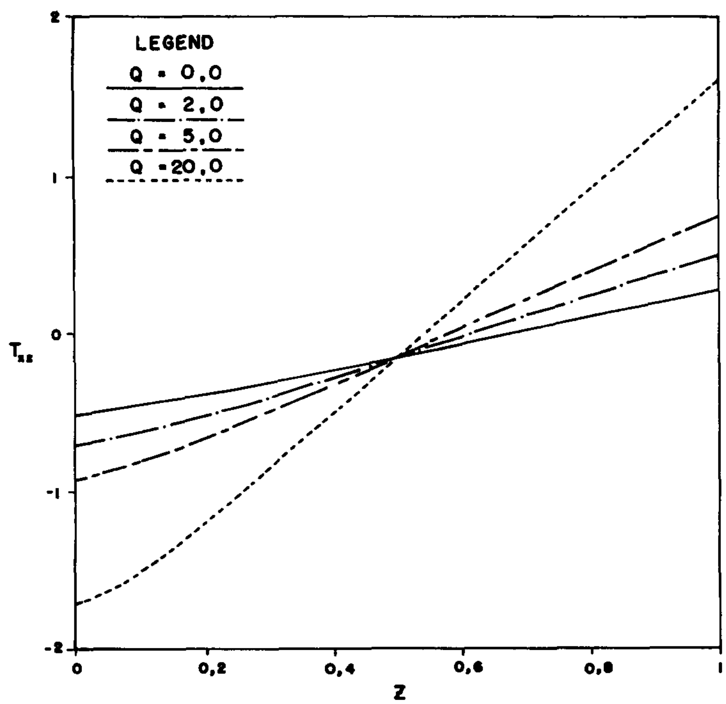

Fig. 3. Stress variation with the temperature parameter Q for problems in which displacement is specified at the slab upper surface. ($K_1 = 1.15$.)

$$T_{XZ} = \frac{(1+Q)f'(1)}{f(1)} = K_2, \tag{51}$$

with K_2 constant, should replace the boundary condition (45) used in the problem discussed before, while equations (43), (44) and (46) remain the same. Figures 4–6 show the results

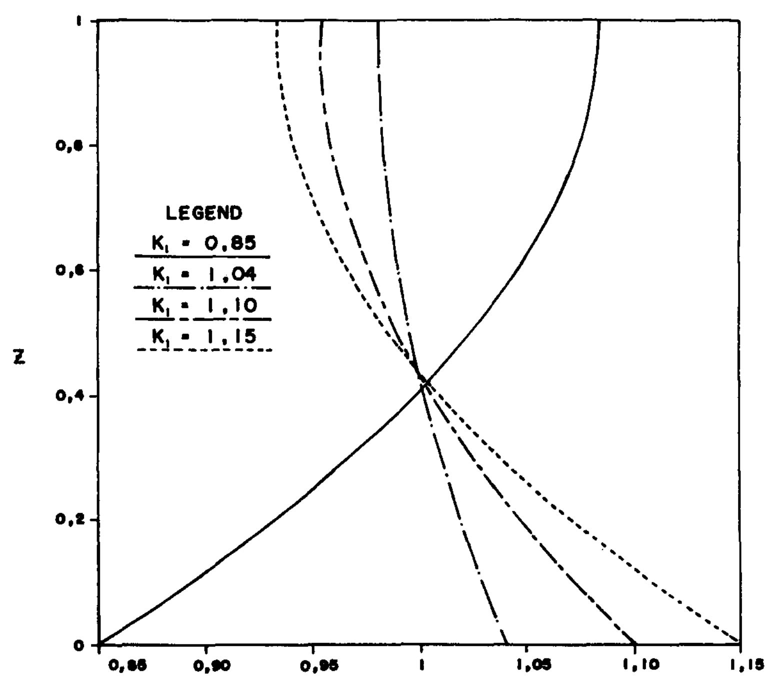

Fig. 4. Variation of f with the shearing constant K_1 for problems in which stress is specified at the slab upper surface. ($Q = 0$.)

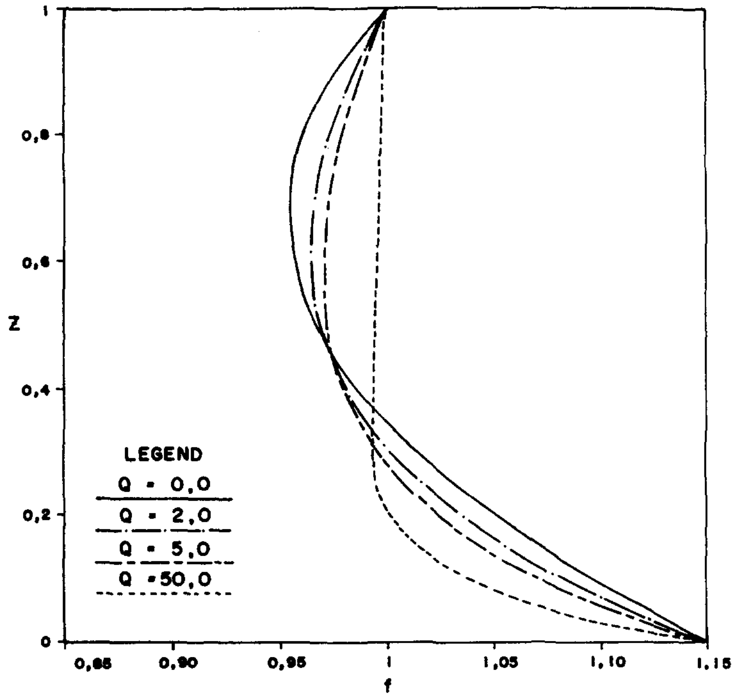

Fig. 5. Variation of f with the temperature parameter Q for problems in which stress is specified at the slab upper surface. ($K_1 = 1.10$.)

found assuming $K_2 = 0$. This corresponds to a problem in which the upper surface of the slab is free to move along the horizontal direction. The displacement function for isothermal conditions and different values of the shearing constant is displayed in Fig. 4. As noted in the previous example, the deformation is nonhomogeneous at all positions across the

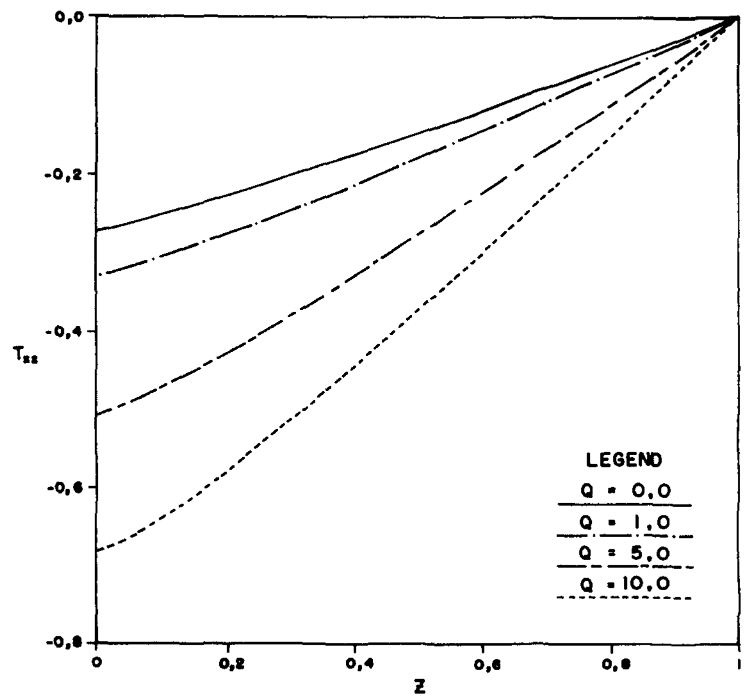

Fig. 6. Stress variation with the temperature parameter Q for problems in which stress is specified at the slab upper surface. ($K_1 = 1.10$.)

plate thickness. However, if the temperature dependence is incorporated into the problem, "boundary layer" type of solution is again observed as the temperature differential increases. This can be seen from Fig. 5. The stress distribution is depicted in Fig. 6 for various values of the temperature gradient and $K_1 = 1.10$. Only negative stresses are developed in the slab as the result of the upper surface being shear stress free.

REFERENCES

Antman, S. and Gou, Z. H. (1984). Large shearing oscillations of incompressible nonlinear elastic bodies. *J. Elasticity* **34,** 126–128.

Carrol, M. M. (1977). Plane circular shearing of incompressible fluids and solids. *Quart. J. Mech. Appl. Mathematics* **30,** 223–234.

Chadwick, P. (1974). Thermomechanical properties of rubberlike materials. *Philosophical Trans. R. Soc.* **276,** 371–403.

Ericksen, J. L. (1954). Deformations possible in every isotropic incompressible perfectly elastic body. *Z. Angew. Math. Phys.* **5,** 466–488.

Ericksen, J. L. (1955). Deformations possible in every isotropic incompressible perfectly elastic material. *J. Math. Phys.* **34,** 126–128.

Eringen, A. C. (1967). *Mechanics of Continua.* Wiley, Chichester.

Haughton, D. M. (1992). Boundary layer solutions for incompressible elastic cylinders. *Int. J. Engng Sci.* **30,** 1027–1040.

Maneschy, C. E., Rajagopal, K. R. and Massoudi, M. (1993). Inhomogeneous deformations in finite thermoelasticity. *Int. J. Engng Sci.* (in press).

Ogden, R. W. (1992). On the thermoelastic modeling of rubberlike solids. *J. Thermal Stresses* **15,** 533–557.

Rajagopal, K. R. and Wineman, A. (1985). New exact solutions in non-linear elasticity. *Int. J. Engng Sci.* **23,** 217–324.

Rajagopal, K. R., Troy, W. C. and Wineman, A. (1986). A note on non-universal deformations of non-linear elastic layers. *Proc. R. Ir. Ac.* **86A,** 107–114.

Rajagopal, K. R. and Tao, L. (1992). On a inhomogeneous deformation of a generalized neo-Hookean material. *J. Elasticity* **28,** 165–184.

Treloar, L. R. G. (1975) *Physics of Rubber Elasticity.* Oxford.

Truesdell, C. and Noll, W. (1965). Non-linear field theories of mechanics. *HandBuch der Physik III/3.* Springer-Verlag, Berlin.